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n -th Derivatives of Some Functions in terms of N-Fractional Calculus

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Abstract

In this article, N-fractional calculus and $n(\in Z^+)$ -th derivatives of functions

$$f(z) = \frac{1}{(\sqrt{z-b}-c)^2-d} \quad ((\sqrt{z-b}-c)^2-d \neq 0)$$

are discussed. That is, n -th derivatives of the function,

$$(f(z))_n = (-1)^n (z-b)^{-1-n} \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k [\frac{k}{2}+1+m]_n}{m!k!} S^k T^m$$

where

$$S = \frac{c}{\sqrt{z-b}}, \quad T = \frac{d}{z-b} \quad (|S| < 1, \quad |T| < 1).$$

is reported for example.

1 Definition of N-Fractional Calculus

In order to treat the derivatives of arbitrary order, we describe the definition of fractional calculus and some basic theorems and identities.

(I) Definition. (by K. Nishimoto, [1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$, C_- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C_+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$\begin{aligned} f_\nu &= (f)_\nu = {}_C(f)_\nu \\ &= \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^{\nu+1}} \quad (\nu \notin Z^-), \end{aligned} \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in Z^+), \quad (2)$$

where

$$-\pi \leq \arg(\zeta - z) \leq \pi \text{ for } C_-, \quad 0 \leq \arg(\zeta - z) \leq 2\pi \text{ for } C_+,$$

$$\zeta \neq z, \quad z \in C, \quad \nu \in R, \quad \Gamma; \text{ Gamma function,}$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν , some fundamental properties have reported. ([3], [5])

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta-z)^{\nu+1}} \right) \quad (\nu \notin Z^-), \quad (\text{Refer to [1]}) \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in Z^+), \quad (4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in R), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu | \nu \in R\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| \leq \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. Fractional calculus operator group $\{N^\nu\}$ is an Action product group which has continuous index ν for the set of F .

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in R). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. ([5])

(III) We have following results for some elementary functions. ([1])

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty)$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c), \quad (|\Gamma(\alpha)| < \infty)$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii),

(iv)

$$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k. \quad (u = u(z), v = v(z))$$

2 Preliminary

(I) The following theorem is reported by K. Nishimoto [12].

Theorem D. We have

(i)

$$(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta-\gamma} \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k! \Gamma(\beta k - \alpha\beta)} \left(\frac{c}{(z-b)^\beta}\right)^k \quad (1)$$

$$(|\frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)}| < \infty),$$

and

(ii)

$$(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta-n} \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} \left(\frac{c}{(z-b)^\beta}\right)^k$$

$$(n \in Z_0^+, \quad \left|\frac{c}{(z-b)^\beta}\right| < 1), \quad (2)$$

where

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \text{with} \quad [\lambda]_0 = 1,$$

(Pochhammer's Notation).

(II) The following theorem is reported by K. Nishimoto already[13].

Theorem E. We have

(i)

$$(((z-b)^\beta - c)^\alpha - d)^\delta_\gamma = e^{-i\pi\gamma} (z-b)^{\alpha\beta\delta-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k \Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{m! k! \Gamma(\beta k - \alpha\beta(\delta-m))} \left(\frac{c}{(z-b)^\beta}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m,$$

$$(|\frac{\Gamma(\beta k - \alpha\beta(\delta-m) + \gamma)}{\Gamma(\beta k - \alpha\beta(\delta-m))}| < \infty), \quad (3)$$

and

(ii)

$$(((z-b)^\beta - c)^\alpha - d)^\delta_n = (-1)^n (z-b)^{\alpha\beta\delta-n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\delta]_m [-\alpha(\delta-m)]_k [\beta k - \alpha\beta(\delta-m)]_n}{m! k!} \left(\frac{c}{(z-b)^\beta}\right)^k \left(\frac{d}{(z-b)^{\alpha\beta}}\right)^m,$$

$$(n \in Z_0^+) \quad (4)$$

where

$$((z-b)^\beta - c)^\alpha - d \neq 0, \quad \left|\frac{c}{(z-b)^\beta}\right| < 1, \quad \left|\frac{d}{(z-b)^{\alpha\beta}}\right| < 1).$$

We apply these theorems to obtain some theorems for the function

$$\frac{1}{(\sqrt{z-b-c})^2-d}.$$

3 N-Fractional Calculus of Functions $\frac{1}{(\sqrt{z-b}-c)^2-d}$

Theorem 1. We have

(i)

$$\begin{aligned} \left(\frac{1}{(\sqrt{z-b}-c)^2-d} \right)_\gamma &= e^{-i\pi\gamma} (z-b)^{-1-\gamma} \\ &\times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k \Gamma(\frac{k}{2} + 1 + m + \gamma)}{m! k! \Gamma(\frac{k}{2} + 1 + m)} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m \quad (1) \\ &(|\Gamma(\frac{k}{2} + 1 + m + \gamma)| < \infty) \end{aligned}$$

and

(ii)

$$\begin{aligned} \left(\frac{1}{(\sqrt{z-b}-c)^2-d} \right)_n &= (-1)^n (z-b)^{-1-n} \\ &\times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k [\frac{k}{2} + 1 + m]_n}{m! k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m \quad (2) \\ &(n \in \mathbb{Z}_0^+) \end{aligned}$$

$$\text{where } (\sqrt{z-b}-c)^2-d \neq 0, \quad \left| \frac{c}{\sqrt{z-b}} \right| < 1, \quad \left| \frac{d}{z-b} \right| < 1.$$

$$\text{and } \sum_{m,k=0}^{\infty} := \sum_{m=0}^{\infty} \sum_{k=0}^{\infty}.$$

Proof of (i). We operate N^γ to the both sides of following relation,

$$\frac{1}{(\sqrt{z-b}-c)^2-d} = \left(((z-b)^{1/2}-c)^2-d \right)^{-1} \quad (3)$$

and by setting $\alpha = 2$, $\beta = 1/2$, $\delta = -1$ in Theorem E (i) we obtain (1), under the conditions stated before.

Proof of (ii). We have the result by setting $\gamma = n$ in the equation (1).

Furthermore by setting $\alpha = 0$ in Theorem 1, we have the following corollary immediately.

Corollary 1. We have

(i)

$$\left(\frac{1}{z - 2c\sqrt{z-b} - b + c^2} \right)_\gamma = e^{-i\pi\gamma} (z-b)^{-1-\gamma} \sum_{k=0}^{\infty} \frac{[2]_k \Gamma(\frac{k}{2} + 1 + \gamma)}{k! \Gamma(\frac{k}{2} + 1)} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (4)$$

$$(|\Gamma(\frac{k}{2} + 1 + \gamma)| \leq \infty)$$

and

(ii)

$$\left(\frac{1}{z - 2c\sqrt{z-b} - b + c^2} \right)_n = (-1)^n (z-b)^{-1-n} \sum_{k=0}^{\infty} \frac{[2]_k [\frac{k}{2} + 1]}{k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (5)$$

$$(n \in \mathbb{Z}_0^+),$$

$$\text{where } \sqrt{z-b} - c \neq 0, \quad \left| \frac{c}{\sqrt{z-b}} \right| \leq 1.$$

Theorem 2. We have the following identities,

(i)

$$(z-b)^{-1/2} \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k \Gamma(\frac{k}{2} + 1 + m + \gamma)}{m! k! \Gamma(\frac{k}{2} + 1 + m)} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m$$

$$= \frac{1}{2\sqrt{d}} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(\frac{k}{2} + \frac{1}{2} + \gamma)}{k! \Gamma(\frac{k}{2} + \frac{1}{2})} \left\{ \left(\frac{c + \sqrt{d}}{\sqrt{z-b}} \right)^k - \left(\frac{c - \sqrt{d}}{\sqrt{z-b}} \right)^k \right\} \quad (6)$$

$$(|\Gamma(\frac{k}{2} + 1 + m + \gamma)| < \infty) \quad (|\Gamma(\frac{k}{2} + \frac{1}{2} + \gamma)| < \infty)$$

and

(ii) for $n \in \mathbb{Z}_0^+$,

$$(z-b)^{-1/2} \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k [\frac{k}{2} + 1 + m]_n}{m! k!} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m$$

$$= \frac{1}{2\sqrt{d}} \sum_{k=0}^{\infty} \frac{[1]_k [\frac{k}{2} + \frac{1}{2}]_n}{k!} \left\{ \left(\frac{c + \sqrt{d}}{\sqrt{z-b}} \right)^k - \left(\frac{c - \sqrt{d}}{\sqrt{z-b}} \right)^k \right\} \quad (7)$$

where $(\sqrt{z-b} - c)^2 - d \neq 0, \quad d \neq 0,$

$$\left| \frac{c}{\sqrt{z-b}} \right| \neq 1, \quad \left| \frac{d}{z-b} \right| \neq 1, \quad \left| \frac{c + \sqrt{d}}{\sqrt{z-b}} \right| \neq 1, \quad \left| \frac{c - \sqrt{d}}{\sqrt{z-b}} \right| \neq 1.$$

Proof of (i). We have the following relation,

$$\frac{1}{(\sqrt{z-b}-c)^2-d} = \frac{1}{2\sqrt{d}} \left(\frac{1}{\sqrt{z-b}-c-\sqrt{d}} - \frac{1}{\sqrt{z-b}-c+\sqrt{d}} \right). \quad (8)$$

From Theorem D,(i), we have

$$\left(\frac{1}{\sqrt{z-b}-p} \right)_\gamma = \left(((z-b)^{1/2}-p)^{-1} \right)_\gamma \quad (9)$$

$$= e^{-i\pi\gamma} (z-b)^{-1/2-\gamma} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(\frac{k}{2} + \frac{1}{2} + \gamma)}{k! \Gamma(\frac{k}{2} + \frac{1}{2})} \left(\frac{p}{\sqrt{z-b}} \right)^k. \quad (10)$$

$$(|\Gamma(2m+2+\gamma-k)| < \infty)$$

Therefore, setting $p = c + \sqrt{d}$ or $c - \sqrt{d}$, we have

$$\begin{aligned} \left(\frac{1}{(\sqrt{z-b}-c)^2-d} \right)_\gamma &= \frac{1}{2\sqrt{d}} \left(\left(\frac{1}{\sqrt{z-b}-c-\sqrt{d}} \right)_\gamma - \left(\frac{1}{\sqrt{z-b}-c+\sqrt{d}} \right)_\gamma \right) \\ &= e^{-i\pi\gamma} (z-b)^{-1/2-\gamma} \sum_{k=0}^{\infty} \frac{[1]_k \Gamma(\frac{k}{2} + \frac{1}{2} + \gamma)}{k! \Gamma(\frac{k}{2} + \frac{1}{2})} \left\{ \left(\frac{c+\sqrt{d}}{\sqrt{z-b}} \right)^k - \left(\frac{c-\sqrt{d}}{\sqrt{z-b}} \right)^k \right\}. \end{aligned} \quad (11)$$

Proof of (ii). Set $\gamma = n$ in (6).

4 Semi Derivatives and Integrals

[I] We have

(i)

$$\begin{aligned} \left(\frac{1}{(\sqrt{z-b}-c)^2-d} \right)_{1/2} &= -i(z-b)^{-3/2} \\ &\times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k \Gamma(\frac{k}{2} + \frac{3}{2} + m)}{m! k! \Gamma(\frac{k}{2} + 1 + m)} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m \end{aligned} \quad (1)$$

(semi derivative)

and

(ii)

$$\left(\frac{1}{(\sqrt{z-b}-c)^2-d} \right)_{-1/2} = i(z-b)^{-1/2} \times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k \Gamma(\frac{k}{2} + \frac{1}{2} + m)}{m! k! \Gamma(\frac{k}{2} + 1 + m)} \left(\frac{c}{\sqrt{z-b}} \right)^k \left(\frac{d}{z-b} \right)^m \quad (2)$$

(semi integral)

where

$$(\sqrt{z-b}-c)^2-d \neq 0, \quad \left| \frac{c}{\sqrt{z-b}} \right| < 1, \quad \left| \frac{d}{z-b} \right| < 1,$$

from Theorem 1 by setting $\gamma = 1/2$ and $-1/2$, respectively.

[II] We have

(i)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2} \right)_{1/2} = -i(z-b)^{-3/2} \times \sum_{k=0}^{\infty} \frac{[2]_k \Gamma(\frac{k}{2} + \frac{3}{2})}{k! \Gamma(\frac{k}{2} + 1)} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (3)$$

(semi derivative)

and

(ii)

$$\left(\frac{1}{z-2c\sqrt{z-b}-b+c^2} \right)_{-1/2} = i(z-b)^{-1/2} \times \sum_{k=0}^{\infty} \frac{[2]_k \Gamma(\frac{k}{2} + \frac{1}{2})}{k! \Gamma(\frac{k}{2} + 1)} \left(\frac{c}{\sqrt{z-b}} \right)^k \quad (4)$$

(semi integral)

where

$$\left| \frac{c}{\sqrt{z-b}} \right| < 1,$$

from Corollary 1 by setting $\gamma = 1/2$ and $-1/2$, respectively.

5 Some Special Cases

When the order of differentiation is some integer, our results coincide the classical calculus. So, we illustrate some examples in cases of $n = 0, 1$.

(I) When $n = 0$, from Theorem 1.(ii), we have the followings,

$$\left(\frac{1}{(\sqrt{z-b-c})^2 - d} \right)_0 = (z-b)^{-1} \times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k [\frac{k}{2} + 1 + m]_0}{m!k!} S^k T^m \quad (1)$$

Here we set

$$S = \frac{c}{\sqrt{z-b}}, \quad T = \frac{d}{z-b}.$$

Indeed we have

$$R.H.S. \text{ of (1)} = (z-b)^{-1} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[2+2m]_k}{k!} S^k \quad (2)$$

$$= (z-b)^{-1} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m (1-S)^{-2-2m} \quad (3)$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} \left(\frac{T}{(1-S)^2} \right)^m \quad (4)$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2} \left(1 - \frac{T}{(1-S)^2} \right)^{-1} \quad (5)$$

$$= \frac{1}{z-b} \frac{1}{(1-S)^2 - T} = \frac{1}{(\sqrt{z-b-c})^2 - d}. \quad (6)$$

(II) When $n = 1$, our result is written as follows.

$$\left(\frac{1}{(\sqrt{z-b-c})^2 - d} \right)_1 = -(z-b)^{-2} \times \sum_{m,k=0}^{\infty} \frac{[1]_m [2+2m]_k [\frac{k}{2} + 1 + m]_1}{m!k!} S^k T^m. \quad (7)$$

Indeed we have

$$R.H.S. \text{ of (7)} = -(z-b)^{-2} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} T^m \times \sum_{k=0}^{\infty} \frac{[2+2m]_k (\frac{k}{2} + 1 + m)}{k!} S^k. \quad (8)$$

Now we take notice that

$$\sum_{k=0}^{\infty} \frac{[2+2m]_k (\frac{k}{2})}{k!} S^k = \frac{1}{2} \sum_{k=1}^{\infty} \frac{[2+2m]_k}{(k-1)!} S^k \quad (9)$$

$$= \frac{1}{2} S \sum_{k=0}^{\infty} \frac{[2+2m]_{k+1}}{k!} S^k \quad (10)$$

$$= \frac{1}{2} S(2+2m) \sum_{k=0}^{\infty} \frac{[3+2m]_k}{k!} S^k \quad (11)$$

$$= S(1+m)(1-S)^{-3-2m} \quad (12)$$

and

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{[2+2m]_k (1+m)}{k!} S^k &= (1+m) \sum_{k=0}^{\infty} \frac{[2+2m]_k}{k!} S^k \\ &= (1+m)(1-S)^{-2-2m}. \end{aligned} \quad (13)$$

Therefore we obtain

$$\begin{aligned} R.H.S. \text{ of (7)} &= -(z-b)^{-2}(1-S)^{-3} \sum_{m=0}^{\infty} \frac{[1]_m}{m!} (1+m) \left(\frac{T}{(1-S)^2} \right)^m \quad (14) \\ &= -(z-b)^{-2}(1-S)^{-3} \left\{ \sum_{m=0}^{\infty} \frac{[1]_m}{m!} \left(\frac{T}{(1-S)^2} \right)^m + \sum_{m=0}^{\infty} \frac{[1]_m}{m!} \left(\frac{T}{(1-S)^2} \right)^m \right\} \\ &= -(z-b)^{-2}(1-S)^{-3} \left\{ \left(1 - \frac{T}{(1-S)^2} \right)^{-1} + \frac{T}{(1-S)^2} \left(1 - \frac{T}{(1-S)^2} \right)^{-2} \right\} \\ &= -(z-b)^{-2}(1-S)^{-3} \frac{(1-S)^4}{((1-S)^2 - T)^2} \\ &= -(z-b)^{-2} \frac{1-S}{((1-S)^2 - T)^2} \\ &= -\frac{1}{\sqrt{z-b}} \frac{\sqrt{z-b}-c}{((\sqrt{z-b}-c)^2 - d)^2}. \end{aligned} \quad (15)$$

This result coincides with the one obtained from the classical calculus

$$\frac{d}{dz} \left(\frac{1}{(\sqrt{z-b}-c)^2 - d} \right)$$

(III) The cases of $n = 2$ and 3 are somewhat complicated, we will report those cases at another time.

Note. In this section we use the following identities.

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} S^k = (1 - S)^{-\lambda}, \quad (|S| < 1) \quad (16)$$

$$[\lambda]_{k+1} = \lambda[\lambda + 1]_k \quad (17)$$

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k k}{k!} S^k = \sum_{k=1}^{\infty} \frac{[\lambda]_k}{(k-1)!} S^k = S \sum_{k=0}^{\infty} \frac{[\lambda]_{k+1}}{k!} S^k \quad (18)$$

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